A stabilized mixed DG formulation for flow in porous media with double pore-networks

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What problem we are solving?

Macro-pore network | **Micro-pore network**

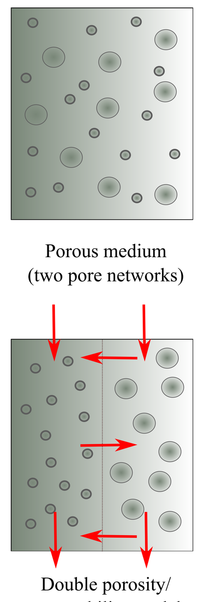
BoLM:
$$\mu \mathbf{K}_1^{-1}\mathbf{u}_1(\mathbf{x}) + \mathrm{grad}[p_1] = \gamma \mathbf{b}(\mathbf{x}) \quad \text{in } \Omega \qquad \mu \mathbf{K}_2^{-1}\mathbf{u}_2(\mathbf{x}) + \mathrm{grad}[p_2] = \gamma \mathbf{b}(\mathbf{x}) \quad \text{in } \Omega$$

BoM:
$$\mathrm{div}[\mathbf{u}_1] = +\chi(\mathbf{x}) \quad \text{in } \Omega \qquad \mathrm{div}[\mathbf{u}_2] = -\chi(\mathbf{x}) \quad \text{in } \Omega$$

V-B.C.:
$$\mathbf{u}_1(\mathbf{x}) \cdot \widehat{\mathbf{n}}(\mathbf{x}) = u_{n1}(\mathbf{x}) \quad \text{on } \Gamma_1^v \qquad \mathbf{u}_2(\mathbf{x}) \cdot \widehat{\mathbf{n}}(\mathbf{x}) = u_{n2}(\mathbf{x}) \quad \text{on } \Gamma_2^v$$

P-B.C.:
$$p_1(\mathbf{x}) = p_{01}(\mathbf{x}) \quad \text{on } \Gamma_1^p \qquad p_2(\mathbf{x}) = p_{02}(\mathbf{x}) \quad \text{on } \Gamma_2^p$$

$$\chi(\mathbf{x}) = -\frac{\beta}{\mu}(p_1(\mathbf{x}) - p_2(\mathbf{x}))$$



Proposed a stabilized mixed DG formulation

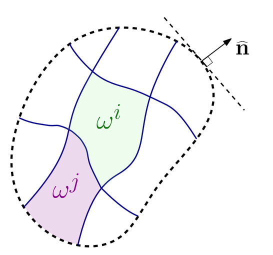
- The classical mixed DG formulation will not be stable under any combination of interpolation functions for the field variables. Specifically, equal-order interpolation for all the field variables (which is computationally the most convenient) is not stable under the classical mixed DG formulation.

The jump and average operators on an interior edge for a scalar field $\varphi(\mathbf{x})$ are, respectively, defined as follows:

$$[\![\varphi]\!] := \varphi^+ \widehat{\mathbf{n}}^+ + \varphi^- \widehat{\mathbf{n}}^- \quad \text{and} \quad \{\!\{\varphi\}\!\} := \frac{\varphi^+ + \varphi^-}{2}$$

where

$$\varphi^+ = \varphi|_{\partial\omega^+} \quad \text{and} \quad \varphi^- = \varphi|_{\partial\omega^-}$$

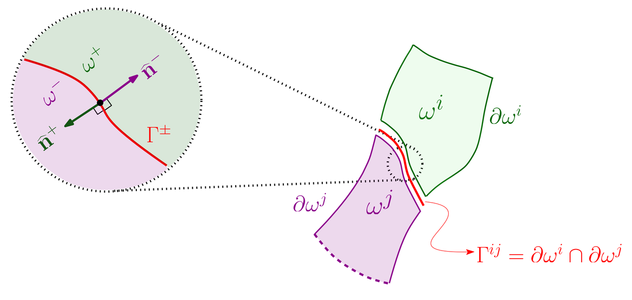


For a vector field $\boldsymbol{\tau}(\mathbf{x})$ these operators are defined as follows:

$$[\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}^+ \cdot \widehat{\mathbf{n}}^+ + \boldsymbol{\tau}^- \cdot \widehat{\mathbf{n}}^- \quad \text{and} \quad \{\!\{\boldsymbol{\tau}\}\!\} := \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2} \quad \text{on } \Gamma^{\mathrm{int}}$$

The following identity will be frequently used:

$$[\![\varphi\boldsymbol{\tau}]\!] = [\![\boldsymbol{\tau}]\!]\{\!\{\varphi\}\!\} + \{\!\{\boldsymbol{\tau}\}\!\} \cdot [\![\varphi]\!]$$



- The corresponding mathematical statement reads as follows: Find $(\mathbf{u}_1(\mathbf{x}), \mathbf{u}_2(\mathbf{x})) \in \mathcal{U} \times \mathcal{U}$, $(p_1(\mathbf{x}), p_2(\mathbf{x})) \in \mathcal{Q}$ such that we have

$$\mathcal{B}_{\mathrm{DG}}^{\mathrm{stab}}(\mathbf{w}_1, \mathbf{w}_2, q_1, q_2; \mathbf{u}_1, \mathbf{u}_2, p_1, p_2) = \mathcal{L}_{\mathrm{DG}}^{\mathrm{stab}}(\mathbf{w}_1, \mathbf{w}_2, q_1, q_2)$$
$$\forall\, (\mathbf{w}_1(\mathbf{x}), \mathbf{w}_2(\mathbf{x})) \in \mathcal{U} \times \mathcal{U}, \;\; (q_1(\mathbf{x}), q_2(\mathbf{x})) \in \mathcal{Q}$$

where the conventional bilinear form and the linear functional are, respectively, defined as follows:

$$\begin{aligned}
\mathcal{B}_{\mathrm{DG}}^{\mathrm{stab}} := \mathcal{B}_{\mathrm{DG}} &- \frac{1}{2}\left(\mu k_1^{-1}\mathbf{w}_1 - \mathrm{grad}[q_1]; \mu^{-1}k_1(\mu k_1^{-1}\mathbf{u}_1 + \mathrm{grad}[p_1])\right) \\
&- \frac{1}{2}\left(\mu k_2^{-1}\mathbf{w}_2 - \mathrm{grad}[q_2]; \mu^{-1}k_2(\mu k_2^{-1}\mathbf{u}_2 + \mathrm{grad}[p_2])\right) \\
&+ \eta_u h \left(\{\!\{\mu k_1^{-1}\}\!\}[\![\mathbf{w}_1]\!]; [\![\mathbf{u}_1]\!]\right)_{\Gamma^{\mathrm{int}}} + \eta_u h \left(\{\!\{\mu k_2^{-1}\}\!\}[\![\mathbf{w}_2]\!]; [\![\mathbf{u}_2]\!]\right)_{\Gamma^{\mathrm{int}}} \\
&+ \frac{\eta_p}{h}\left(\{\!\{\mu^{-1}k_1\}\!\}[\![q_1]\!]; [\![p_1]\!]\right)_{\Gamma^{\mathrm{int}}} + \frac{\eta_p}{h}\left(\{\!\{\mu^{-1}k_2\}\!\}[\![q_2]\!]; [\![p_2]\!]\right)_{\Gamma^{\mathrm{int}}} \\
\mathcal{L}_{\mathrm{DG}}^{\mathrm{stab}} := \mathcal{L}_{\mathrm{DG}} &- \frac{1}{2}\left(\mu k_1^{-1}\mathbf{w}_1 - \mathrm{grad}[q_1]; \mu^{-1}k_1\gamma\mathbf{b}_1\right) - \frac{1}{2}\left(\mu k_2^{-1}\mathbf{w}_2 - \mathrm{grad}[q_2]; \mu^{-1}k_2\gamma\mathbf{b}_2\right)
\end{aligned}$$

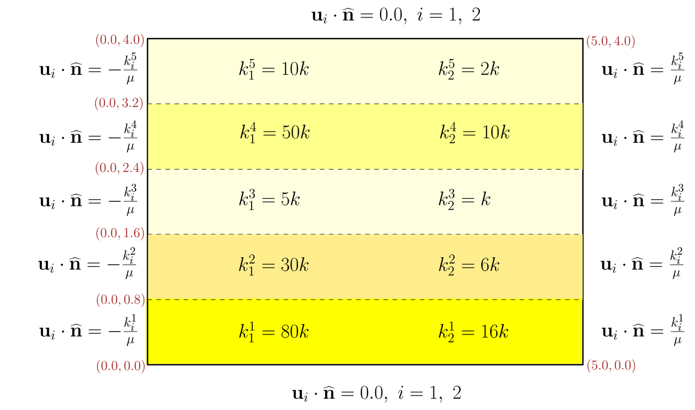
The bilinear form and the linear functional are, respectively, defined as follows:

$$\begin{aligned}
\mathcal{B}_{\mathrm{DG}} := &\left(\mathbf{w}_1; \mu k_1^{-1}\mathbf{u}_1\right) - (\mathrm{div}[\mathbf{w}_1]; p_1) + (q_1; \mathrm{div}[\mathbf{u}_1]) + ([\![\mathbf{w}_1]\!]; \{\!\{p_1\}\!\})_{\Gamma^{\mathrm{int}}} - (\{\!\{q_1\}\!\}; [\![\mathbf{u}_1]\!])_{\Gamma^{\mathrm{int}}} \\
&+ \left(\mathbf{w}_2; \mu k_2^{-1}\mathbf{u}_2\right) - (\mathrm{div}[\mathbf{w}_2]; p_2) + (q_2; \mathrm{div}[\mathbf{u}_2]) + ([\![\mathbf{w}_2]\!]; \{\!\{p_2\}\!\})_{\Gamma^{\mathrm{int}}} - (\{\!\{q_2\}\!\}; [\![\mathbf{u}_2]\!])_{\Gamma^{\mathrm{int}}} \\
&+ \left(q_1 - q_2; \frac{\beta}{\mu}(p_1 - p_2)\right) + (\mathbf{w}_1 \cdot \widehat{\mathbf{n}}; p_1)_{\Gamma_1^u} + (\mathbf{w}_2 \cdot \widehat{\mathbf{n}}; p_2)_{\Gamma_2^u} - (q_1; \mathbf{u}_1 \cdot \widehat{\mathbf{n}})_{\Gamma_1^u} - (q_2; \mathbf{u}_2 \cdot \widehat{\mathbf{n}})_{\Gamma_2^u} \\
\mathcal{L}_{\mathrm{DG}} := &(\mathbf{w}_1; \gamma\mathbf{b}_1) + (\mathbf{w}_2; \gamma\mathbf{b}_2) - (\mathbf{w}_1 \cdot \widehat{\mathbf{n}}; p_{01})_{\Gamma_1^p} - (\mathbf{w}_2 \cdot \widehat{\mathbf{n}}; p_{02})_{\Gamma_2^p} - (q_1; u_{n1})_{\Gamma_1^u} - (q_2; u_{n2})_{\Gamma_2^u}
\end{aligned}$$

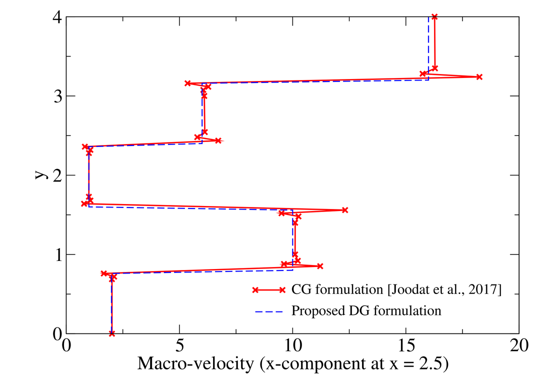
- η_u and η_p are non-negative, non-dimensional numbers. h is the element length scale
- In order to minimize the drift in the solution fields, additional stabilization terms on the internal boundaries (i.e., terms containing u and p) are required in both networks.

patch tests are practically useful

- The proposed formulation is capable of modeling flow in a highly heterogeneous, layered porous domain with abrupt changes in macro- and micro-permeabilities



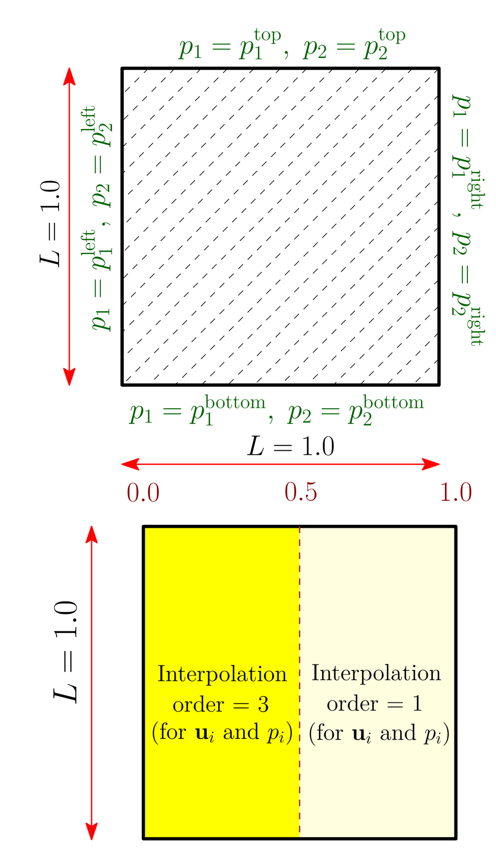
Five-layer velocity-driven patch test boundary conditions, and macro- and micro-permeabilities

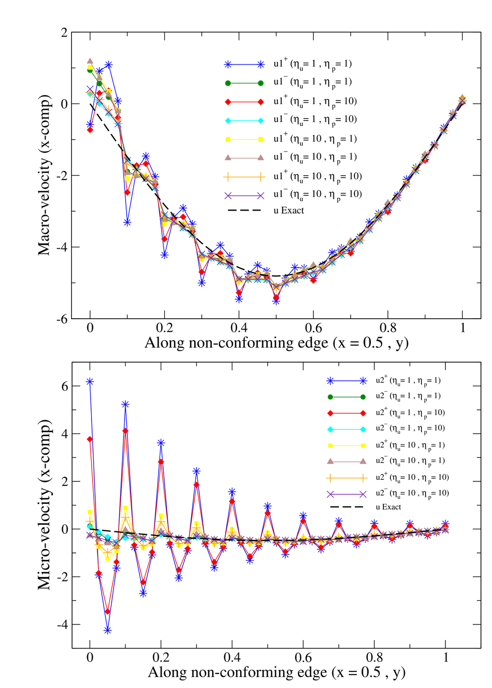


Comparison of the velocity profiles obtained under CG and DG. Under the CG formulation. overshoots and undershoots are observed along the interfaces of the layers

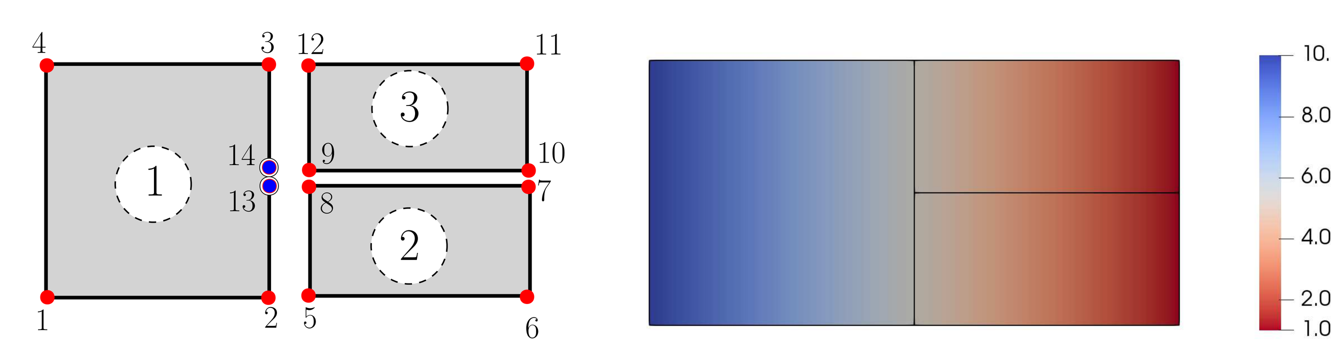
Non-conforming polynomial orders

quantifying the effect of additional stabilization terms on the accuracy of results under double porosity/permeability model for the problems exhibiting mismatching interpolation order.

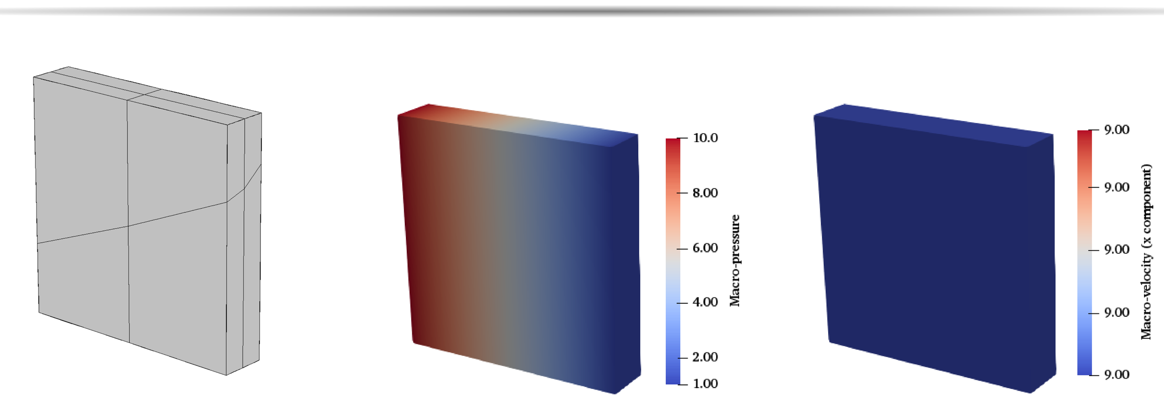




Sensitivity analysis of η_p and η_u



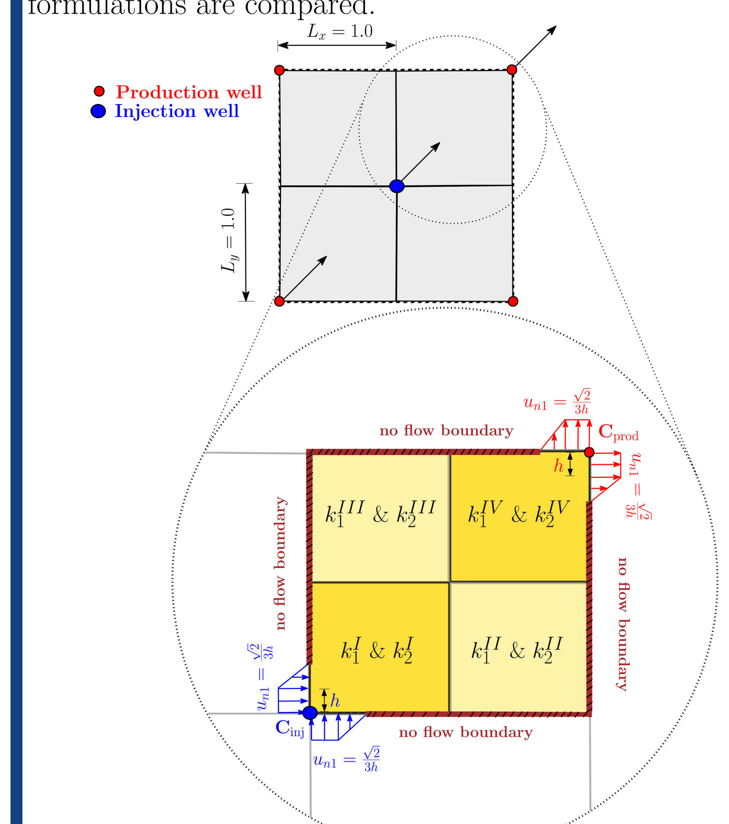
The formulation is capable of handling non-conforming element refinement (hanging node in the mesh).Pressures in both pore-networks are varying linearly as velocities are constant throughout the domain.

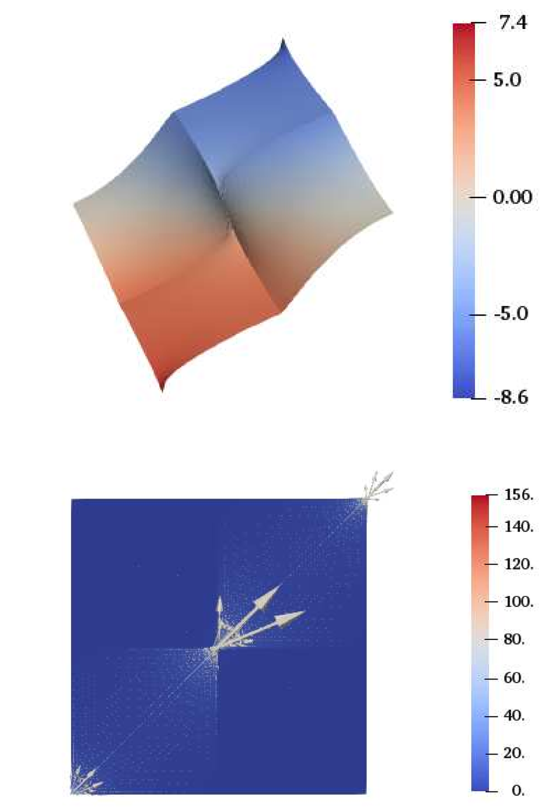


- The formulation is capable of providing accurate results for non-constant Jacobian elements.

The formulation is robust

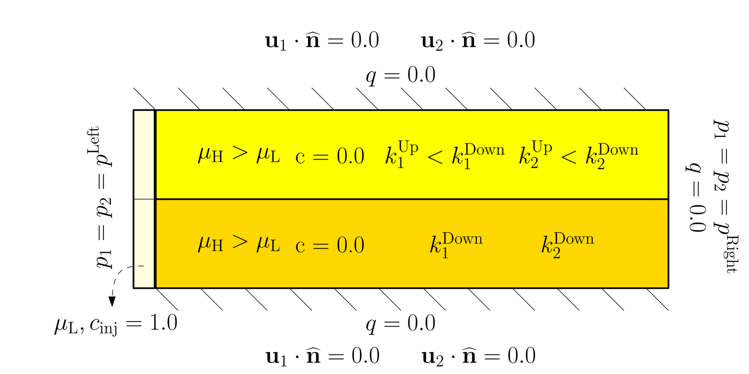
The formulation with abrupt changes in material properties and elliptic singularities, which is typically referred to as *quarter five-spot checkerboard* problem. Moreover, *the element-wise mass balance* property of the CG and DG formulations are compared.

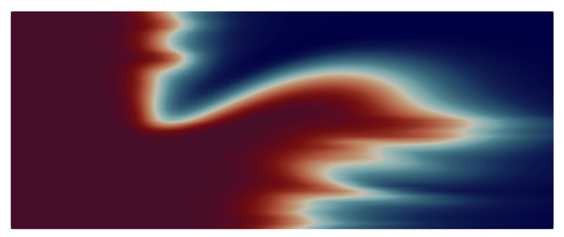




An extension to coupled problems

- Viscous fingering (VF), or the so-called Saffman-Taylor instability, generally refers to the onset and evolution of instabilities that occur in the displacement of fluids in porous media.
- The VF occurs in porous media, when a more viscous fluid is displaced by a less viscous fluid.
- Darcy model coupled with the transport equation can exhibit such instabilities.





Key message

- The proposed formulation is capable of suppressing the non-physical numerical instabilities observed in coupled flow and transport problems, yet capturing the underlying physical instabilities under double porosity/permeability model.

Conclusions

- Computationally convenient equal-order interpolation for all the field variables is stable under the proposed stabilized mixed DG formulation.
- The stabilization terms are residual-based and the stabilization parameters do not contain any mesh-dependent parameters.
- The formulation passes patch tests, even on meshes with non-constant Jacobian elements in 2D and 3D settings. The proposed DG formulation performs remarkably well, in comparison with its continuous counterpart, in the presence of abrupt changes in the medium properties.
- The DG formulation can support non-conforming discretization in form of non-conforming polynomial orders or non-conforming element refinement, thus allowing efficient h-, p-, and hp-adaptivities.
- The sensitivity study of the velocity solutions with respect to the stabilization parameters explains the stronger effect of η_u on reducing the drift for the case of non-conforming polynomial orders.
- It is shown that the proposed DG formulation can be employed to solve coupled flow-transport problems in porous media with double pore-networks. The proposed formulation is capable of suppressing the non-physical numerical instabilities, yet capturing the underlying physical instabilities.