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Composable Block Solvers and Performance Spectrum Model for the Four-Field Double Porosity/Permeability Model

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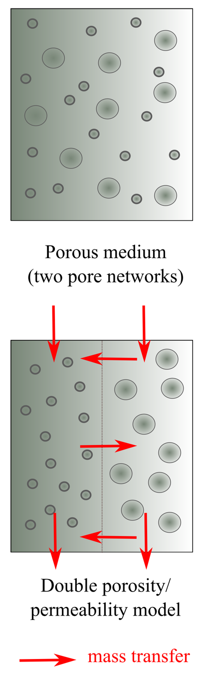
What problem we are solving? DPP

	Macro-pore network	Micro-pore network
BoLM	$\mu \mathbf{K}_1^{-1}\mathbf{u}_1(\mathbf{x}) + \mathrm{grad}[p_1] = \gamma \mathbf{b}(\mathbf{x})$ in Ω	$\mu \mathbf{K}_2^{-1}\mathbf{u}_2(\mathbf{x}) + \mathrm{grad}[p_2] = \gamma \mathbf{b}(\mathbf{x})$ in Ω
BoM	$\mathrm{div}[\mathbf{u}_1] = +\chi(\mathbf{x})$ in Ω	$\mathrm{div}[\mathbf{u}_2] = -\chi(\mathbf{x})$ in Ω
V-B.C.	$\mathbf{u}_1(\mathbf{x}) \cdot \widehat{\mathbf{n}}(\mathbf{x}) = u_{n1}(\mathbf{x})$ on Γ_1^v	$\mathbf{u}_2(\mathbf{x}) \cdot \widehat{\mathbf{n}}(\mathbf{x}) = u_{n2}(\mathbf{x})$ on Γ_2^v
P-B.C.	$p_1(\mathbf{x}) = p_{01}(\mathbf{x})$ on Γ_1^p	$p_2(\mathbf{x}) = p_{02}(\mathbf{x})$ on Γ_2^p

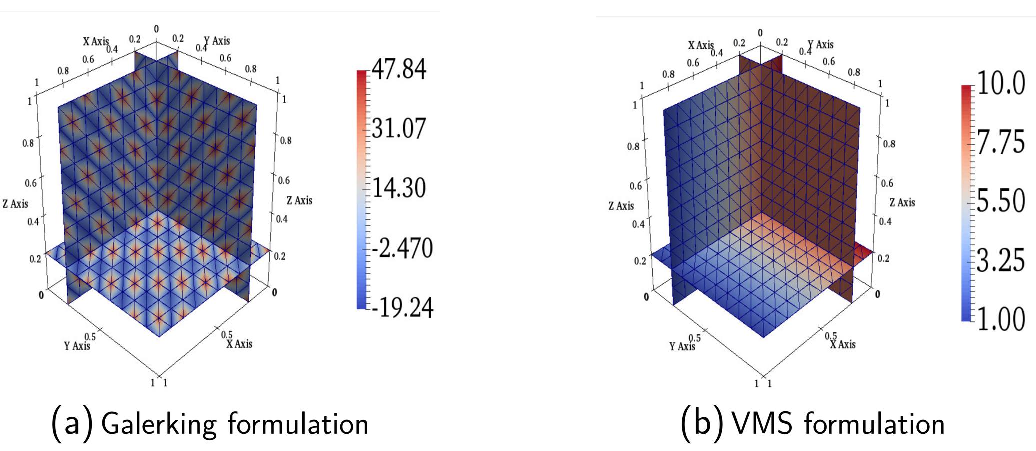
$$\chi(\mathbf{x}) = -\frac{\beta}{\mu}(p_1(\mathbf{x}) - p_2(\mathbf{x}))$$



What are the numerical challenges?

Does CG "the finite element" work for DPP?

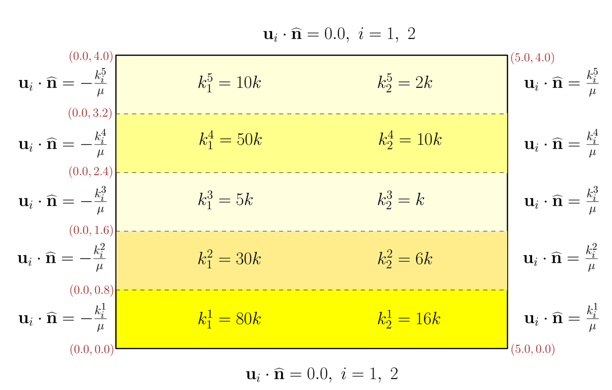
- Not for equal order interpolations of $\mathbf{u}$ and p. [LBB violation]



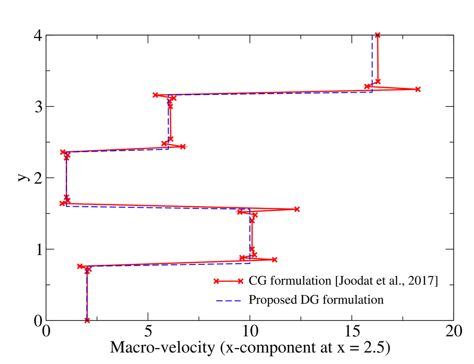
(a) Galerking formulation (b) VMS formulation

Can we use VMS-type formulation in heterogeneous domains?

- CG-VMS results in overshoots and undershoots at the vicinity of material jump[Gibbs phenomenon]



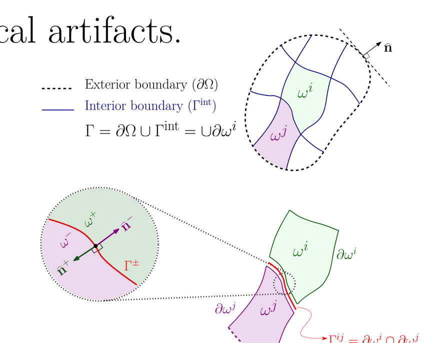
Five-layer velocity-driven patch test bcs, and macro- and micro-permeabilities



Velocity profiles at $x = 0.5$.

- **DG-VMS formulation:** suppresses above numerical artifacts.

$$\mathcal{B}_{\mathrm{DG}}^{\mathrm{stab}}(\mathbf{w}_1, \mathbf{w}_2, q_1, q_2; \mathbf{u}_1, \mathbf{u}_2, p_1, p_2) = \mathcal{L}_{\mathrm{DG}}^{\mathrm{stab}}(\mathbf{w}_1, \mathbf{w}_2, q_1, q_2)$$
$$\forall (\mathbf{w}_1(\mathbf{x}), \mathbf{w}_2(\mathbf{x})) \in \mathcal{U} \times \mathcal{U}, \ (q_1(\mathbf{x}), q_2(\mathbf{x})) \in \mathcal{Q}$$



The bilinear form:

$$\begin{aligned}\mathcal{B}_{\mathrm{DG}}^{\mathrm{stab}} := \mathcal{B}_{\mathrm{DG}} &- \frac{1}{2}\left(\mu k_1^{-1}\mathbf{w}_1 - \mathrm{grad}[q_1]; \mu^{-1}k_1(\mu k_1^{-1}\mathbf{u}_1 + \mathrm{grad}[p_1])\right) \\ &- \frac{1}{2}\left(\mu k_2^{-1}\mathbf{w}_2 - \mathrm{grad}[q_2]; \mu^{-1}k_2(\mu k_2^{-1}\mathbf{u}_2 + \mathrm{grad}[p_2])\right) \\ &+ \eta_u h\left(\{\!\{\mu k_1^{-1}\}\!\}[\![\mathbf{w}_1]\!]; [\![\mathbf{u}_1]\!]\right)_{\Gamma^{\mathrm{int}}} + \eta_u h\left(\{\!\{\mu k_2^{-1}\}\!\}[\![\mathbf{w}_2]\!]; [\![\mathbf{u}_2]\!]\right)_{\Gamma^{\mathrm{int}}} \\ &+ \frac{\eta_p}{h}\left(\{\!\{\mu^{-1}k_1\}\!\}[\![q_1]\!]; [\![p_1]\!]\right)_{\Gamma^{\mathrm{int}}} + \frac{\eta_p}{h}\left(\{\!\{\mu^{-1}k_2\}\!\}[\![q_2]\!]; [\![p_2]\!]\right)_{\Gamma^{\mathrm{int}}}\end{aligned}$$

The linear functional:

$$\mathcal{L}_{\mathrm{DG}}^{\mathrm{stab}} := \mathcal{L}_{\mathrm{DG}} - \frac{1}{2}\left(\mu k_1^{-1}\mathbf{w}_1 - \mathrm{grad}[q_1]; \mu^{-1}k_1\gamma\mathbf{b}_1\right) - \frac{1}{2}\left(\mu k_2^{-1}\mathbf{w}_2 - \mathrm{grad}[q_2]; \mu^{-1}k_2\gamma\mathbf{b}_2\right)$$

- η_u and η_p are non-negative, non-dimensional stabilization parameters.

What composable block solvers we proposed?

Discrete formulations for the DPP model can be assembled into: $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$

Here are two ways to effectively precondition our large system of equations:

Method 1: splitting by scales

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_{uu}^1 & \boldsymbol{K}_{up}^1 & 0 & 0 \\ \boldsymbol{K}_{pu}^1 & \boldsymbol{K}_{pp}^1 & 0 & \boldsymbol{K}_{pp}^{12} \\ 0 & 0 & \boldsymbol{K}_{uu}^2 & \boldsymbol{K}_{up}^2 \\ 0 & \boldsymbol{K}_{pp}^{21} & \boldsymbol{K}_{pu}^2 & \boldsymbol{K}_{pp}^2 \end{bmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{p}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{p}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_u^1 \\ \boldsymbol{f}_p^1 \\ \boldsymbol{f}_u^2 \\ \boldsymbol{f}_p^2 \end{pmatrix} \qquad \boldsymbol{A} := \begin{bmatrix} \boldsymbol{K}_{uu}^1 & \boldsymbol{K}_{up}^1 \\ \boldsymbol{K}_{pu}^1 & \boldsymbol{K}_{pp}^1 \end{bmatrix}, \quad \boldsymbol{B} := \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{K}_{pp}^{12} \end{bmatrix}, \quad \boldsymbol{C} := \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{K}_{pp}^{21} \end{bmatrix}, \quad \boldsymbol{D} := \begin{bmatrix} \boldsymbol{K}_{uu}^2 & \boldsymbol{K}_{up}^2 \\ \boldsymbol{K}_{pu}^2 & \boldsymbol{K}_{pp}^2 \end{bmatrix}$$

- $\boldsymbol{A}$ and $\boldsymbol{D}$ have similar compositions to classical mixed Poisson → Schur complement approach.
- Individually precondition the decoupled $\boldsymbol{A}$ and $\boldsymbol{D}$ blocks.

The inverses of $\boldsymbol{A}$, and $\boldsymbol{B}$ is:

$$\boldsymbol{A}^{-1} = \begin{bmatrix} \boldsymbol{I} & -(\boldsymbol{K}_{uu}^1)^{-1}\boldsymbol{K}_{up}^1 \\ 0 & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} (\boldsymbol{K}_{uu}^1)^{-1} & 0 \\ 0 & (\boldsymbol{S}^1)^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & 0 \\ -\boldsymbol{K}_{pu}^1(\boldsymbol{K}_{uu}^1)^{-1} & \boldsymbol{I} \end{bmatrix}$$
$$\boldsymbol{D}^{-1} = \begin{bmatrix} \boldsymbol{I} & -(\boldsymbol{K}_{uu}^2)^{-1}\boldsymbol{K}_{up}^2 \\ 0 & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} (\boldsymbol{K}_{uu}^2)^{-1} & 0 \\ 0 & (\boldsymbol{S}^2)^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & 0 \\ -\boldsymbol{K}_{pu}^2(\boldsymbol{K}_{uu}^2)^{-1} & \boldsymbol{I} \end{bmatrix}$$

1. $\boldsymbol{K}_{uu}^1$, $\boldsymbol{K}_{uu}^2$ are mass matrices → ILU(0) to invert.
2. Precondition $\boldsymbol{S}^1$, and $\boldsymbol{S}^2$:
$$\boldsymbol{S}_p^1 = \boldsymbol{K}_{pp}^1 - \boldsymbol{K}_{pu}^1 \mathrm{diag}(\boldsymbol{K}_{uu}^1)^{-1}\boldsymbol{K}_{up}^1$$
$$\boldsymbol{S}_p^2 = \boldsymbol{K}_{pp}^2 - \boldsymbol{K}_{pu}^2 \mathrm{diag}(\boldsymbol{K}_{uu}^2)^{-1}\boldsymbol{K}_{up}^2$$
3. Apply multigrid V-cycle on $\boldsymbol{S}_p^1$ and $\boldsymbol{S}_p^2$ from the HYPRE BoomerAMG.
4. Single sweep of flexible GMRES to obtain the solution of full 4×4 block system.

Method 2: splitting by fields

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_{uu}^1 & 0 & \boldsymbol{K}_{up}^1 & 0 \\ 0 & \boldsymbol{K}_{uu}^2 & 0 & \boldsymbol{K}_{up}^2 \\ \boldsymbol{K}_{pu}^1 & 0 & \boldsymbol{K}_{pp}^1 & \boldsymbol{K}_{pp}^{12} \\ 0 & \boldsymbol{K}_{pu}^2 & \boldsymbol{K}_{pp}^{21} & \boldsymbol{K}_{pp}^2 \end{bmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_u^1 \\ \boldsymbol{f}_u^2 \\ \boldsymbol{f}_p^1 \\ \boldsymbol{f}_p^2 \end{pmatrix} \qquad \boldsymbol{A} := \begin{bmatrix} \boldsymbol{K}_{uu}^1 & 0 \\ 0 & \boldsymbol{K}_{uu}^2 \end{bmatrix}, \quad \boldsymbol{B} := \begin{bmatrix} \boldsymbol{K}_{up}^1 & 0 \\ 0 & \boldsymbol{K}_{up}^2 \end{bmatrix}, \quad \boldsymbol{C} := \begin{bmatrix} \boldsymbol{K}_{pu}^1 & 0 \\ 0 & \boldsymbol{K}_{pu}^2 \end{bmatrix}, \quad \boldsymbol{D} := \begin{bmatrix} \boldsymbol{K}_{pp}^1 & \boldsymbol{K}_{pp}^{12} \\ \boldsymbol{K}_{pp}^{21} & \boldsymbol{K}_{pp}^2 \end{bmatrix}$$

- The inverse of $\mathbf{K}$ is:

$$\boldsymbol{K}^{-1} = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{A}^{-1}\boldsymbol{B} \\ 0 & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}^{-1} & 0 \\ 0 & \boldsymbol{S}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & 0 \\ -\boldsymbol{C}\boldsymbol{A}^{-1} & \boldsymbol{I} \end{bmatrix}.$$

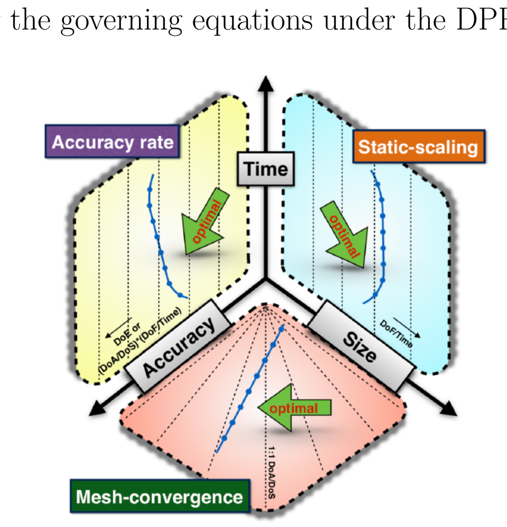
1. $\boldsymbol{A}$ is mass matrix → use ILU(0).
2. Precondition $\boldsymbol{S}^{-1}$ by employing diagonal mass-lumping of $\boldsymbol{A}$:
$$\boldsymbol{S}_p = \boldsymbol{D} - \boldsymbol{C}\mathrm{diag}(\boldsymbol{A})^{-1}\boldsymbol{B} = \begin{bmatrix} \boldsymbol{K}_{pp}^1 - \boldsymbol{K}_{pu}^1 \mathrm{diag}(\boldsymbol{K}_{uu}^1)\boldsymbol{K}_{up}^1 & \boldsymbol{K}_{pp}^{12} \\ \boldsymbol{K}_{pp}^{21} & \boldsymbol{K}_{pp}^2 - \boldsymbol{K}_{pu}^2 \mathrm{diag}(\boldsymbol{K}_{uu}^2)\boldsymbol{K}_{up}^2 \end{bmatrix}$$
3. Diag blocks similar to $\boldsymbol{S}_p^1$ and $\boldsymbol{S}_p^2$ → use V-cycle on each.
4. Single sweep of flexible GMRES to obtain the solution of full system.

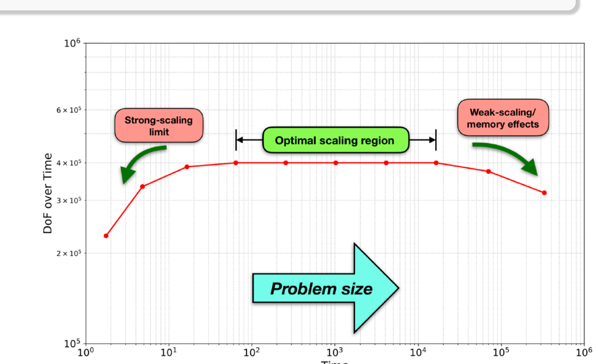
We used the composable solvers feature in PETSc and the finite element libraries under the Firedrake Project.

What is performance spectrum model?

(i) Which of the 3 FEs (CG-VMS, DG-VMS, or H(div)) performs better for **large-scale** DPP?
(ii) Which of the proposed solver performs better for **large-scale** DPP?

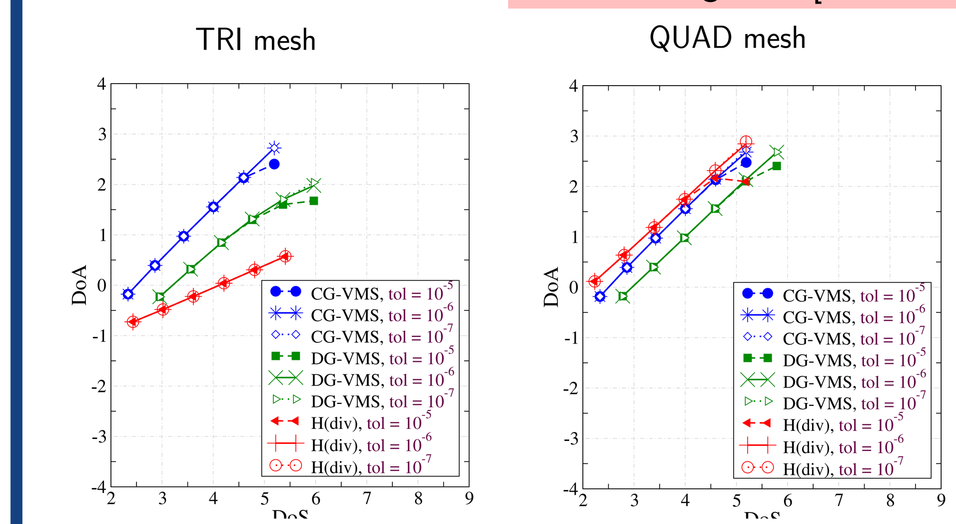
- We compare the performance of the chosen three FEs for solving the governing equations under the DPP model.





- Parallel efficiency $= \frac{Time_1}{Time_p \times \# MPI processors}\%$
- *Digits of Accuracy* DoA:$= -\log_{10}(L_2^{\mathrm{norm}})$
  Digits of Size DoS:$= -\log_{10}(DoF)$
- *Digits of Efficacy* DoE:$= -\log_{10}(L_2^{\mathrm{norm}} \times Time)$

Mesh convergence: [Macro-velocity]



- Tight solver tol is needed to avoid flattening out.
- Same pattern obsereved for $\mathbf{u}_2$,p_1, and p_2.

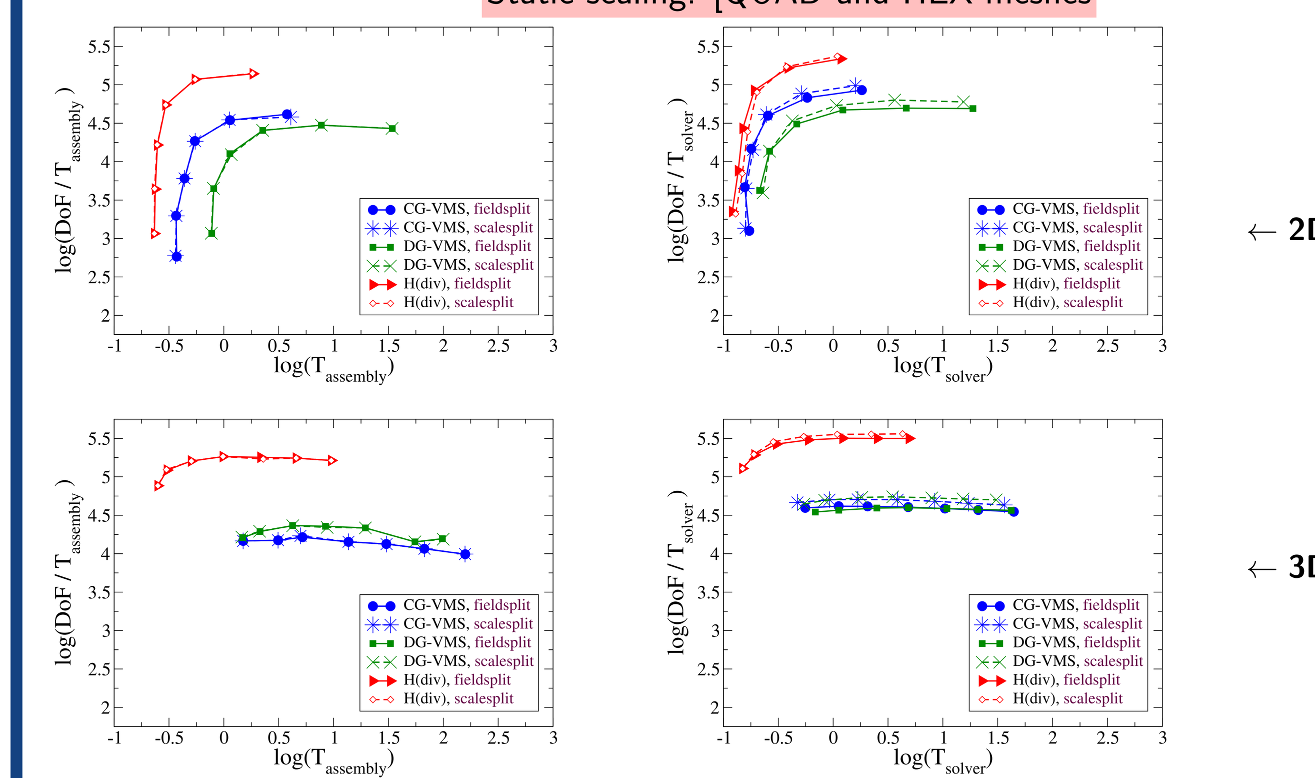
For TRI:

- VMS slope ≈ 1. H(div) slop ≈ 0.5
- Numerical accuracy: VMS > H(div)

For QUAD:

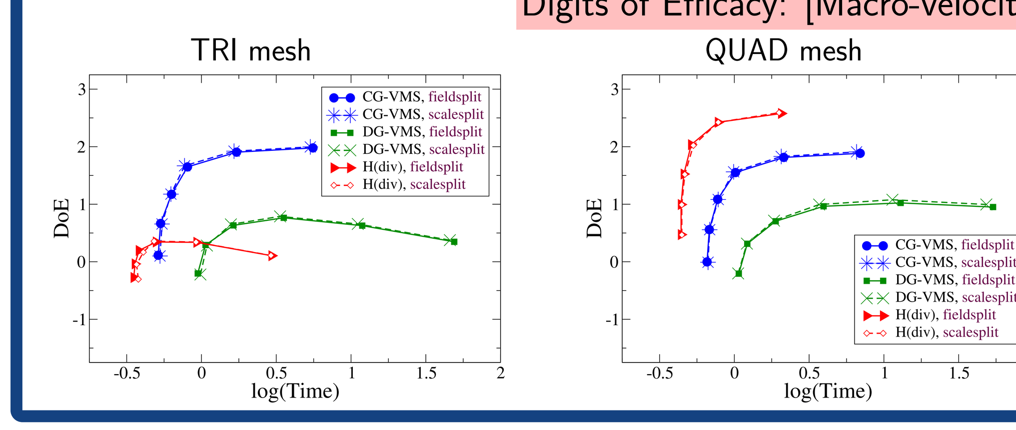
- VMS slope ≈ 1.
- H(div) shows super. conv. close ≈ 1.
- Numerical accuracy: VMS ≳ H(div).

Static scaling: [QUAD and HEX meshes]



- H(div) processes its DoF count faster than either of the VMS formulations.
- Assembly algorithm in 2D vs 3D

Digits of Efficacy: [Macro-velocity]



For TRI:

- DoE: CG-VMS ≫ DG-VMS > H(div).

For QUAD:

- DoE: H(div) > CG-VMS > DG-VMS.

Conclusions

- Proposed a framework for performance analysis of various "enriched FEs" for the DPP model.
- The VMS formulations yield much higher overall numerical accuracy for all velocity and pressure fields.
- Regardless of mesh type, DoFs are processed the fastest under the H(div) formulation compared to other formulations.
- Both composable solvers are scalable in both parallel and algorithmic senses.
- Both solvers exert similar overall effects on performance metrics.

References

[1] M. S. Joshaghani, S. H. S. Joodat, and K. B. Nakshatrala. A stabilized mixed discontinuous galerkin formulation for double porosity/permeability model. *arXiv preprint arXiv:1805.01389*, 2018.

[2] M. S. Joshaghani, J. Chang, K. B. Nakshatrala, and M. G. Knepley. Composable block solvers for the four-field double porosity/permeability model. *arXiv preprint arXiv:1808.08328*, 2018.

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